THE STRUCTURE OF STRONG LINEAR PRESERVERS OF GW-MAJORIZATION ON $\mathrm{M}_{N,M}$*

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Abstract. Let $\mathbf{M}_{n,m}$ be the set of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. A matrix $R \in M_n$ with the property Re=e, is said to be a g-row stochastic (generalized row stochastic) matrix. Let A,B$\in$ $\mathbf{M}_{n,m}$, so B is said to be gw-majorized by A if there exists an n×n g-row stochastic matrix R such that B=RA. In this paper we characterize all linear operators that strongly preserve gw-majorization on $\mathbf{M}_{n,m}$ and all linear operators that strongly preserve matrix majorization on $\mathbf{M}_n$.

Key words. Preserver, strong preserver, g-row stochastic matrices, gw-majorization.

AMS subject classifications. 15A03, 15A04, 15A51.

1. Introduction. A nonnegative matrix $R \in M_n$ with the property Re=e, is said to be a row stochastic matrix. Let A,B$\in$ $\mathbf{M}_{n,m}$, so B is said to be matrix-majorized by A if there exists an n×n row stochastic matrix R such that B=RA. The definition of matrix majorization was introduced by Dahl in [6]. For more information about majorization see [5] and [8].

Let $\sim$ be a relation on $\mathbf{M}_{n,m}$. A linear operator T:$\mathbf{M}_{n,m} \longrightarrow \mathbf{M}_{n,m}$ is said to be a linear strong preserver of $\sim$ whenever:

$$x \sim y \quad \Longleftrightarrow \quad T(x) \sim T(y).$$

A matrix $D \in M_n$ with the properties De=e and D^te=e, is said to be a g-doubly stochastic matrix. Let A,B$\in$ $\mathbf{M}_{n,m}$, so B is said to be gs-majorized by A if there exists an n×n g-doubly stochastic matrix D such that B=DA. The definition of gs-majorization was introduced in [1] and authors proved that a linear operator $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ strongly preserves gs-majorization if and only if $T(X) = AXR + JXS$ for some $R, S \in \mathbf{M}_m$ and $A \in \mathbf{M}_n$, such that A, R and $R + nS$ are invertible and A is g-doubly stochastic .

In [3], Beasley, S.-G. Lee and Y.H Lee proved that, if a linear operator T:$\mathbf{M}_n \to \mathbf{M}_n$ strongly preserves matrix majorization then, there exist a permutation P and an invertible matrix $M \in \mathbf{M}_n$ such that $T(X) = PXM$ for every X in $span\{\mathbf{R}_n\}$, where $\mathbf{R}_n$ is the set of all $n \times n$ row stochastic matrices, and currently A.M. Hasani and M. Radjabalipour in [7] showed that:

$$T(X) = PXM, \forall\ X \in\ \mathbf{M}_n. \tag{1.1}$$

In [2] authors introduced gw-majorization and characterized its strong linear preservers on $\mathbf{M}_n$. In this paper, we will to show that a linear operator $T : \mathbf{M}_{n,m} \to$

*Received by the editors on ... Accepted for publication on .... Handling Editor: ...

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$\mathbf{M}_{n,m}$ strongly preserves gw-majorization if and only if $T(X) = AXB$ for every X in $\mathbf{M}_{n,m}$, where $A \in \mathbf{GR}_n$ and $B \in \mathbf{M}_m$ are invertible matrices. In the end we state a corollary that regains (1.1).

Throughout this paper, $\mathbf{GR}_n$ is the set of all g-row stochastic matrices , e=$(1, ..., 1)^t \in \mathbb{F}^n$ and J=$ee^t \in \mathbf{M}_n$.

2. Strong linear preservers of gw-majorization on $\mathbf{M}_{n,m}$. In this section we state some properties of gw-majorization on $\mathbf{M}_{n,m}$ then we characterize all linear operators on $\mathbf{M}_{n,m}$ that strongly preserve gw-majorization.

A matrix $R \in \mathbf{M}_n$ with the property Re=e, is said to be a g-row stochastic matrix. For more details see [4].

DEFINITION 2.1. *Let $A, B \in \mathbf{M}_{n,m}$. The matrix B is said to be gw-majorized by A if there exists an $n \times n$ g-row stochastic matrix R such that B=RA and denoted by $A \succ_{gw} B$.*

PROPOSITION 2.2. *Let $T : \mathbf{M}_{n,m} \rightarrow \mathbf{M}_{n,m}$ be a linear operator that strongly preserves gw-majorization . Then T is invertible.*

Proof. Suppose T(A)=0. Since T is linear and $0 \succ_{gw}$T(A), T(0) $\succ_{gw}$T(A). Therefore, $0 \succ_{gw}$A because T strongly preserves gw-majorization. Then, there exists an n×n g-row stochastic matrix R such that A=R0. Then, A=0 and hence T is invertible. □

REMARK 2.3. *Let A,B be two g-row stochastic matrices then, AB and A^{-1} (If A is invertible) are g-row stochastic matrices.*

The relation gw-majorization on $\mathbf{M}_{n,m}$ has the following properties :
Let $X, Y \in \mathbf{M}_{n,m}$, $A, B \in \mathbf{GR}_n$, $C \in \mathbf{M}_m$ and $\alpha, \beta \in \mathbb{F}$ such that A,B and C are invertible and $\alpha \neq 0$. Then the following conditions are equivalent:

1. $X \succ_{gw} Y$
2. $AX \succ_{gw} BY$
3. $\alpha X + \beta J_{n,m} \succ_{gw} \alpha Y + \beta J_{n,m}$
4. $XC \succ_{gw} YC$

Where $J_{n,m}$ is the $n \times m$ matrix whose all entries are equal one.

Now, we characterize the linear preservers of gw-majorization on $\mathbb{F}^n$.

LEMMA 2.4. *Let $x \in \mathbb{F}^n$. Then $x \succ_{gw} y$, $\forall y \in \mathbb{F}^n$ if and only if $x \notin span\{e\}$.*

Proof. Let $x \succ_{gw} y$, $\forall y \in \mathbb{F}^n$, it is clear that $x \notin span\{e\}$. Conversely, let $x = (x_1, \cdots, x_n)^t \notin span\{e\}$, then x has at least two distinct components such as x_k and x_l. Let $y = (y_1, \cdots, y_n)^t \in \mathbb{F}^n$ be arbitrary, for $1 \leq i, j \leq n$ define

$r_{ik} = \frac{y_i - x_l}{x_k - x_l}$, $r_{il} = \frac{-y_i + x_k}{x_k - x_l}$ and $r_{ij} = 0$ If $j \neq k, l$. Then $R = (r_{ij}) \in \mathbf{GR}_n$ and $Rx = y$, so $x \succ_{gw} y$. □

LEMMA 2.5. *Let $T : \mathbb{F}^n \rightarrow \mathbb{F}^n$ be a non zero linear operator. Then T preserves gw-majorization if and only if $x \notin span\{e\}$ implies that $T(x) \notin span\{e\}$.*

Proof. Let T preserves gw-majorization. Assume that $x \notin span\{e\}$, then $x \succ_{gw} y$, $\forall y \in \mathbb{F}^n$ by Lemma 2.4. Therefore $T(x) \succ_{gw} T(y)$, $\forall y \in \mathbb{F}^n$. If $T(x) \in span\{e\}$ then T=0, a contradiction, so $T(x) \notin span\{e\}$.

Conversely, let $x \notin span\{e\}$ implies that $T(x) \notin span\{e\}$. If $x \succ_{gw} y$ then we have two cases:

Case 1; Let $x \in span\{e\}$, then x=y and hence T(x)=T(y).

Case 2; Let $x \notin span\{e\}$, then $T(x) \notin span\{e\}$ by hypostasis, so by Lemma 2.4, $T(x) \succ_{gw} Z, \forall Z \in \mathbb{F}^n$ and hence $T(x) \succ_{gw} T(y)$. Then T preserves gw-majorization. □

THEOREM 2.6. *Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T preserves gw-majorization if and only if $T(x) = \alpha Rx$ for some $R \in \mathbf{M}_n$ and $\alpha \in \mathbb{F}$, such that either $ker(R) = span\{e\}$ and $e \notin Im(R)$ or $R \in \mathbf{GR}_n$ is invertible.*

Proof. If T =0, we put $\alpha = 0$. Let $T \neq 0$ and A be the matrix representation of T with respect to the standard basis of $\mathbb{F}^n$. Now, we consider two cases:

Case 1; Let T be invertible. Then there exits $b \in \mathbb{F}^n$ such that Ab=e. So b=re, for some $r \in \mathbb{F}$, by Lemma 2.5. Then $Ae = \frac{1}{r}e$, therefore $T(x) = \alpha Rx$, where $\alpha = \frac{1}{r}$ and $R = (rA) \in \mathbf{GR}_n$ is invertible.

Case 2; Let T be singular. Then by Lemma 2.5, ker(T)= span$\{e\}$ and $e \notin Im(T)$. So ker(A)=span$\{e\}$ and $e \notin Im(A)$. The converse is trivial. □

Now, we state the following two Lemmas to prove the main Theorem of this paper.

LEMMA 2.7. *Let $A \in \boldsymbol{M}_n$ be such that ker(A)=span$\{e\}$. Then there exist $x_0, y_0 \in \mathbb{F}^n$ and $R_0 \in \mathbf{GR}_n$ such that $x_0 + Ay_0$ doesn't gw-majorize $R_0x_0 + AR_0y_0$.*

Proof. Assume if possible,

$$x + Ay \succ_{gw} Rx + ARy, \forall x, y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n \quad . \tag{2.1}$$

Now, we consider two cases:

Case 1; Let $e \in Im(A)$, then there exists $y_0 \in \mathbb{F}^n$, such that $Ay_0 = e$. Put $x = 0, y = y_0$ in (2.1) then $ARy_0 = e, \forall R \in \mathbf{GR}_n$, a contradiction .

Case 2; Let $e \notin Im(A)$, then $\mathbb{F}^n = Im(A) \bigoplus span\{e\}$. So for every i $(1 \leq i \leq n)$, there exist $y_i \in \mathbb{F}^n$ and $r_i \in \mathbb{F}$ such that $e_i = Ay_i + r_ie$. Put $x = e - (e_i - r_ie)$ and $y = y_i$ in (2.1), then

$$r_ie - Re_i + ARy_i = 0, \ \forall R \in \mathbf{GR}_n \quad . \tag{2.2}$$

For every j $(1 \leq j \leq n, j \neq i)$ put $R_j = ee_j^t$ in (2.2), then $r_i = 0$, for every i $(1 \leq i \leq n)$. Therefore $Ay_i = e_i$, for every i $(1 \leq i \leq n)$, then $Im(A) = \mathbb{F}^n$, a contradiction. □

LEMMA 2.8. *Let A$\in$ $\mathbf{GR}_n$ be invertible. Then the following conditions are equivalent:*

(a) A=I

(b) $(x + Ay) \succ_{gw} (Rx + ARy)$, $\forall R \in \mathbf{GR}_n$ and $\forall x, y \in \mathbb{F}^n$.

Proof. It is clear that, (a) implies (b). Conversely, let (b) holds. The matrix A is invertible, then for every i $(1 \leq i \leq n)$ there exists $y_i \in \mathbb{F}^n$ such that $Ay_i = e - e_i$. By hypostasis $(e_i + Ay_i) \succ_{gw} (Re_i + ARy_i)$, $\forall R \in \mathbf{GR}_n$, then

$$(Re_i + ARy_i) = e, \ \forall R \in \mathbf{GR}_n \quad . \tag{2.3}$$

For every $R \in \mathbf{GR}_n$, it is clear that $R[J - (n-1)A] \in \mathbf{GR}_n$, therefore by (2.3),

$$\begin{aligned} R[J-(n-1)A]e_i + AR[J-(n-1)A]y_i = e &\Rightarrow (RA - AR)e_i = 0, \ \forall \ i \in \{1, ..., n\} \\ &\Rightarrow AR = RA, \forall \ R \in \mathbf{GR}_n. \end{aligned}$$

So it is easy to show that A=I. □

Now, we state the main Theorem of this paper.

THEOREM 2.9. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then T strongly preserves gw-majorization if and only if $T(X) = AXB$ for every $X \in \mathbf{M}_{n,m}$, where $A \in \mathbf{GR}_n$ and $B \in \boldsymbol{M}_m$ are invertible .*

Proof. If m=1, the result is implied by Theorem 2.6, so let $m \geq 2$. Define the embedding E^j:$\mathbb{F}^n \to \mathbf{M}_{n,m}$ by $E^j(x) = xe_j^t$ and projection E_i :$\mathbf{M}_{n,m} \to \mathbb{F}^n$ by $E_i(X) = Xe_i$ for every $i, j \in \{1, ..., m\}$. Put $T_i^j = E_i T E^j$ and let $X = [x_1| \cdots |x_m] \in \mathbf{M}_{n,m}$ where x_i is the i^{th} column of X . Then,

$$T(X) = T([x_1| \cdots |x_m]) = [\sum_{j=1}^{m} T_1^j(x_j)| \cdots | \sum_{j=1}^{m} T_m^j(x_j)].$$

It is easy to show that T_i^j:$\mathbb{F}^n \to \mathbb{F}^n$ preserves gw-majorizaton. Then by Theorem 2.6, there exist $\alpha_i^j \in \mathbb{F}$, and $A_i^j \in \mathbf{M}_n$ such that $T_i^j(x) = \alpha_i^j A_i^j x$ where either $A_i^j \in \mathbf{GR}_n$ is invertible or $ker(A_i^j) = span\{e\}$ and $e \notin Im(A_i^j)$. Then,

$$T(X) = [\sum_{j=1}^{m} \alpha_i^j A_i^j x_j| \cdots | \sum_{j=1}^{m} \alpha_m^j A_m^j x_j]. \tag{2.4}$$

Now, we consider three steps for the proof.

Step 1. In this step we will to show that, if there exist p and q ($1 \leq p, q \leq m$) such that $\alpha_p^q \neq 0$ and $A_p^q \in \mathbf{GR}_n$ is invertible, then for every j ($1 \leq j \leq m$), $A_p^j = A_p^q$. If $\alpha_p^j = 0$, without lose of generality we can choose $A_p^j = A_p^q$. Let $\alpha_p^j \neq 0$. For every $x, y \in \mathbb{F}^n$, put $X = xe_q^t + ye_j^t$, then $T(X) \succ_{gw} T(RX), \forall R \in \mathbf{GR}_n$ and hence by (2.4),

$$\alpha_p^q A_p^q x + \alpha_p^j A_p^j y \succ_{gw} \alpha_p^q A_p^q Rx + \alpha_p^j A_p^j Ry, \forall x, y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n \Rightarrow$$

$$x + (A_p^q)^{-1} A_p^j (\frac{\alpha_p^j}{\alpha_p^q} y) \succ_{gw} Rx + (A_p^q)^{-1} A_p^j R(\frac{\alpha_p^j}{\alpha_p^q} y), \forall x, y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n \Rightarrow$$

$$x + (A_p^q)^{-1} A_p^j y \succ_{gw} Rx + (A_p^q)^{-1} A_p^j Ry, \forall x, y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n.$$

So by Lemma 2.7, A_p^j is invertible and hence by Lemma 2.8, $A_p^j = A_p^q$. Set $A_p = A_p^q$, then

$$T(X) = [\sum_{j=1}^{m} \alpha_1^j A_1^j x_j| \cdots |A_p \sum_{j=1}^{m} \alpha_p^j x_j| \cdots | \sum_{j=1}^{m} \alpha_m^j A_m^j x_j].$$

Step 2. In this step we will to show that for every i and j ($1 \leq i, j \leq m$), $A_i^j \in \mathbf{GR}_n$ is invertible if $\alpha_i^j \neq 0$. Assume if possible there exist r and s ($1 \leq r, s \leq m$), such that $ker(A_r^s) = span\{e\}$ and $\alpha_r^s \neq 0$. Without lose of generality we can assume that r=m, then by step 1, for every $1 \leq j \leq m$, $ker(A_m^j) = span\{e\}$. Now, we construct a non

zero $n \times m$ matrix U, such that T(U)=0. Consider the vectors:

$$b_1 = \begin{pmatrix} \alpha_1^1 \\ \vdots \\ \alpha_{m-1}^1 \end{pmatrix}, \cdots, b_m = \begin{pmatrix} \alpha_1^m \\ \vdots \\ \alpha_{m-1}^m \end{pmatrix} \in \mathbb{F}^{m-1}.$$

It is clear that $\{b_1, \cdots, b_m\}$ is a linearly dependent set in $\mathbb{F}^{m-1}$, so there exist (not all zero) $\lambda_1, \cdots, \lambda_m \in \mathbb{F}$, such that

$$\sum_{j=1}^{m} \lambda_j \alpha_i^j = 0 \ , \ \forall \ i \in \{1, ..., m-1\}.$$

Now, define $U := [\lambda_1 e| \cdots |\lambda_m e] \in \mathbf{M}_{n,m}$. It is clear that, $U \neq 0$ and

$$T(U) = [\sum_{j=1}^{m} \lambda_j \alpha_1^j A_1^j e| \cdots | \sum_{j=1}^{m} \lambda_j \alpha_m^j A_m^j e].$$

We will show that T(U)=0. Since $ker(A_m^j) = span\{e\}$, it is clear that $\sum_{j=1}^{m} \lambda_j \alpha_m^j A_m^j e$ =0 and hence the last column of T(U) is zero. Now, for every k $(1 \leq k \leq m-1)$, we consider the k^{th} column of T(U):

Case 1; Let $\alpha_k^l \neq 0$ and $A_k^l \in \mathbf{GR}_n$ be invertible for some l $(1 \leq l \leq m)$, then by step 1 ,

$$\sum_{j=1}^{m} \lambda_j \alpha_k^j A_k^j e = A_k^l (\sum_{j=1}^{m} \lambda_j \alpha_k^j) e = 0.$$

Case 2; Let for every j $(1 \leq j \leq m)$, A_k^j be non invertible, then $ker(A_k^j) = span\{e\}$, so $\sum_{j=1}^{m} \lambda_j \alpha_k^j A_k^j e = 0$. Therefor T(U)=0, a contradiction. So by step 1 there exist invertible matrices $A_i \in \mathbf{GR}_n$ $(1 \leq i \leq m)$ such that $T(X) = T[x_1| \cdots |x_m]$ $= [A_1 X a_1| \cdots |A_m X a_m]$, where $a_i = (\alpha_i^1, \cdots, \alpha_i^m)^t$, for every i $(1 \leq i \leq m)$.

Step 3. In this step we will to show that $A_i = A_1$, for all $1 \leq i \leq m$. Now, we show that $rank[a_1|...|a_m] \geq 2$. Assume if possible, $\{a_1, ..., a_m\} \subseteq span\{a\}$, for some $a \in \mathbb{F}^m$. Since $m \geq 2$, then we choose $b \in (span\{a\})^{\perp} \setminus \{0\}$. Define $X_0 := e_1 b^t \in \mathbf{M}_{n,m}$. It is clear that $X_0 \neq 0$ and $T(X_0) = 0$, a contradiction and hence $rank[a_1|...|a_m] \geq 2$. Without lose of generality we can assume that $\{a_1, a_2\}$ is a linearly independent set. Let $X \in \mathbf{M}_{n,m}$ and $R \in \mathbf{GR}_n$ be arbitrary, then

$$\begin{aligned} X \succ_{gw} RX &\Rightarrow T(X) \succ_{gw} T(RX) \\ &\Rightarrow [A_1 X a_1|...|A_m X a_m] \succ_{gw} [A_1 R X a_1|...|A_m R X a_m] \\ &\Rightarrow A_1 X a_1 + A_2 X a_2 \succ_{gw} A_1 R X a_1 + A_2 R X a_2 \\ &\Rightarrow X a_1 + (A_1^{-1} A_2) X a_2 \succ_{gw} R X a_1 + (A_1^{-1} A_2) R X a_2 \ . \end{aligned} \tag{2.5}$$

Since $\{a_1, a_2\}$ is linearly independent, then for every $x, y \in \mathbb{F}^n$, there exits $B_{x,y} \in \mathbf{M}_{n,m}$ such that, $B_{x,y} \ a_1 = x, \quad B_{x,y} \ a_2 = y$, put $X = B_{x,y}$ in (2.5) thus,

$$\begin{aligned} &B_{x,y} \ a_1 + (A_1^{-1} \ A_2) \ B_{x,y} \ a_2 \succ_{gw} R B_{x,y} \ a_2 + (A_1^{-1} \ A_2) \ R B_{x,y} \ a_2 \Rightarrow \\ &\qquad x + (A_1^{-1} A_2) \ y \succ_{gw} Rx + (A_1^{-1} A_2) R y \ , \forall R \in \mathbf{GR}_n \ . \end{aligned}$$

Then by Lemma 2.8, $A_1^{-1}A_2 = I$ and hence $A_2 = A_1$. For every i $(3 \leq i \leq m)$, if $a_i = 0$ we can replace A_i by A_1. If $a_i \neq 0$, then $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is a linearly independent set. By the same method as above, $A_i = A_1$ or $A_i = A_2$. Let $A = A_1$ and hence $A_i = A$ for every i $(1 \leq i \leq m)$. Therefore,

$$T(X) = [AXa_1 \mid \cdots \mid AXa_m] = AXB,$$

where $B = [a_1 \mid \cdots \mid a_m]$ is an invertible matrix in $\mathbf{M}_m$.

Conversely, if T(X)=AXB where $A \in \mathbf{GR}_n$ and $B \in \mathbf{M}_m$ are invertible matrices, it is trivial that T strongly preserves gw-majorization. □

The following statement shows that every strong linear preserver of matrix majorization is an strong linear preserver of gw-majorization but the converse is false.

Proposition 2.10. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator that strongly preserves matrix majorization. Then T strongly preserves gw-majorization .*

Proof. Let $A \succ_{gw} B$. Then there exists a g-row stochastic matrix R such that B=RA. For the g-row stochastic matrix R, there exist scalars $r_1, ..., r_k$ and row stochastic matrices $R_1, ..., R_k$ such that $\sum_{i=1}^{k} r_i = 1$ and $R = \sum_{i=1}^{k} r_i R_i$. For every i $(1 \leq i \leq k)$, $A \succ R_i A$ and hence $T(A) \succ T(R_i A)$. Then there exist row stochastic matrices S_i $(1 \leq i \leq k)$, such that $T(R_i A) = S_i T(A)$. Put S=$\sum_{i=1}^{k} r_i S_i$, it is clear that S is a g-row stochastic matrix and $T(B) = ST(A)$. Therefore $T(A) \succ_{gw} T(B)$. For other side replace T by T^{-1} and similarly conclude that $A \succ_{gw} B$ where $T(A) \succ_{gw} T(B)$. Then T strongly preserves gw-majorization. □

Example 2.11. *Let the linear operator $T : \mathbf{M}_2 \to \mathbf{M}_2$ be such that T(X)=AX, where $A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. It is clear that T strongly preserves gw-majorization by Theorem 2.9 . But T doesn't strongly preserve matrix majorization. For this cosider the following matrices:*

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Now we state the following corollary that characterize all linear operator that strongly preserve matrix majorization on $\mathbf{M}_n$.

Corollary 2.12 (Theorem 5.2 ,7). *A linear operator $T : \mathbf{M}_n \to \mathbf{M}_n$ strongly preserves matrix majorization $\succ$ if and only if $T(X) = PXL$, where P is permutation and $L \in \boldsymbol{M}_n$ is invertible .*

Proof. Let T strongly preserves matrix majorization. Then T strongly preserves gw-majorization by Proposition 2.10. Therefore in view of Theorem 2.9 there exist invertible matrices $A \in \mathbf{GR}_n$ and $B \in \mathbf{M}_n$ such that $T(X) = AXB$ for all $X \in \mathbf{M}_n$. For every row stochastic matrix R, it is clear that $I \succ R$. So $T(I) \succ T(R)$ for every row stochastic matrix R. Then $AIB \succ ARB$ and hence RA^{-1} is a row stochastic matrix, for every row stochastic matrix R. It is easy to show that A^{-1} is a row stochastic matrix. Similarly A is a row stochastic matrix too and hence A is a permutation matrix. □

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